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One-dimensional adiabatic flow of equilibrium gas–particle mixtures in long vertical ducts with friction

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The equations of the steady, adiabatic, one-dimensional flow of an equilibrium mixture of a perfect gas and incompressible particles, in constant-area ducts with friction, are derived taking into account the effects of gravity and of the finite volume of the particles. As is the case for a pure gas, the mixture is shown to be subject to the phenomenon of choking, and the possibility of an adiabatic heating of the mixture in a subsonic expansion is also theoretically predicted for certain flow inlet conditions. The model may be used to approximately describe the conditions existing in portions of volcanic conduits during the Plinian phases of explosive eruptions. Some results of the numerical integration of the equations for a typical application of this type are briefly discussed, thus showing the potential of the model for carrying out rapid analyses of the influence of the main geometrical and flow parameters describing the problem. A non-volcanological application is also analysed to illustrate the possibility of the adiabatic heating of the mixture.

1. Introduction

The flow of gas–particle mixtures in ducts is of interest for many engineering problems, and its analysis has been carried out by means of mathematical models of different complexity according to the specific application (see e.g. Soo 1967; Wallis 1969; Boothroyd 1971; and the reviews by Rudinger 1976 and Crowe 1982).

The range of the flow inlet conditions (in terms of pressure and temperature), the geometry of the system, and the nature, amount, size and distribution of the particles conveyed by the gas, determine the applicability of the assumptions that characterize the various models. Thus both one-dimensional and two-dimensional treatments may be used, while the particles may be considered to be in thermomechanical equilibrium with the gas, or allowed to have a slip velocity and a temperature jump with respect to it. Particle interaction is normally neglected (although its effects have been taken into account by Doss 1985), while the volume of the particles has been shown by Rudinger (1965, 1970) to significantly influence the results when either the ratio of the gas density to the particle density, or the loading (i.e. the ratio between the mass flow of the particles and that of the gas) or both are sufficiently large.

When compressible flows (whose main technical application has so far been the design of nozzles for solid rocket propulsion) are considered, friction at the duct walls is only rarely taken into account, while gravity forces are almost always neglected.

However, a very peculiar geophysical application exists in which neither wall friction nor gravity forces may be neglected, viz. the analysis of the vertical flow of gas–particle mixtures in long volcanic conduits during the so-called Plinian phases of certain explosive eruptions.

These flows originate from the presence in the magmas of high quantities of dissolved volatiles, which, following the decompression caused by the initial opening of a fissure in the crust, give rise to a characteristic eruptive phenomenology (see e.g. Walker 1981; Wilson, Sparks & Walker 1980). In particular, at a certain depth in the conduit bubbles nucleate and then grow until their further growth is inhibited by neighbouring bubbles, and disruption of the magma takes place (Sparks 1978). Above the disruption surface the magmatic fluid has turned into a gas conveying small particles, which may be liquid or partially solid, but which may be assumed to be incompressible. The size of these particles varies inversely with the intensity of the eruption; indeed, in a powerful one their mean dimension may be of the order of 0.1 mm or even less (Wilson *et al.* 1980).

The duration of these eruptive phases is typically of the order of hours, after which the supply pressure decreases and different scenarios may occur according to the particular situation. For instance, we may have the outflow of liquid magma, or phreato-magmatic phenomena involving explosive interactions between magmatic fluid and subterranean waters (Barberi *et al.* 1989). However, during the main part of the Plinian phases, the flow conditions may be regarded as slowly varying in time, so that a quasi-steady analysis may be appropriate.

The modelling of all the details of the eruptive mechanisms is certainly a very difficult task, particularly if it is considered that many of the parameters characterizing the problem (like the geometry and friction coefficient of the conduit, the composition of the magmatic fluid and its initial conditions, etc.) may be estimated only with a large degree of uncertainty, and may vary during the eruption. However, even a simplified mathematical model capable of yielding predictions of the main physical quantities (like pressure distribution along the conduit, mass outflow in unit time, and velocity of the erupting fluid) as a function of the variation, in plausible ranges, of the above-mentioned parameters, might be of considerable help in the interpretation of the geological evidence. This is why several authors have tried to model the thermogasdynamic behaviour of the gas–particle mixtures flowing in volcanic conduits during Plinian eruptions, thus contributing to one important chapter of 'Geological Fluid Mechanics', as Huppert (1986) called the application of fluid mechanics to problems in the geological sciences.

For instance, by neglecting gravity, wall friction and volume of the particles, Kieffer (1982) was able to reduce the problem to the study of the one-dimensional isentropic flow of a 'pseudogas', which behaves like a perfect gas having characteristics dependent on the loading ratio of the mixture; thus all the results of classical inviscid gasdynamics could be used. More specifically, by modelling the volcanic conduit as a long constant-section or slowly converging duct, with a much shorter rapidly diverging final crater, the mass outflow could easily be predicted, because the large ratios between the supply and the exit pressure assure the occurrence of 'choked' flow, with sonic conditions at the beginning of the crater. However, owing to the neglect of wall friction and of gravity (whose effect, as will be shown, is additive to that of friction in a vertical upward motion) this model leads to considerable overestimates of the mass outflow. Furthermore, if inlet pressures of hundreds of bars and high loading ratios are to be considered, the volume of the particles must be taken into account in order to avoid unrealistically high values of the density of the mixture at the depths of the disruption surface (hundreds or thousands of meters); as a consequence, the gas–particle mixture can no longer be treated as a pseudogas.

Wilson *et al.* (1980) and Wilson & Head (1981) did take into account wall friction, gravity forces, finite volume of the particles and variability of the cross-section; however, they solved the problem by considering only the continuity and the momentum equations, and disregarding the energy equation thanks to the assumption of isothermal flow of the mixture under all conditions, i.e. for any geometry of the conduit and loading ratio. In principle, this assumption seems to be less plausible than that which is made in the present work, i.e. that the good thermal contact between the gas and the highly fragmentated particles assures that the temperature of both phases is constantly the same, and that the mixture flows adiabatically through the duct. Indeed, it will be shown that for constant-area flows the temperature variation along the duct is very small if the loading ratio is sufficiently high. However, the same may not be necessarily true for varying-area conduits allowing expansions from subsonic to supersonic flow. Furthermore, the purpose of the present work is the derivation of a rigorous thermogasdynamic model which should be applicable to the flow of mixtures of a gas and small incompressible particles with any loading ratio, from the lowest extreme corresponding to a pure gas, up to the maximum value for which the particles are still fluidized, and discontinuities of the mixture may be neglected.

In the following sections the characteristics and the thermodynamic properties of homogeneous gas–particle mixtures are first considered, and the conditions of applicability of the assumption of thermomechanical equilibrium are briefly discussed. The equations of the steady one-dimensional flow of an equilibrium mixture in a constant-area duct with friction are then derived, taking into account the effects of gravity and of the finite volume of the particles. After a critical analysis of the expressions giving the variation of the flow parameters along the duct, some results of the numerical solution of the equations of motion for various types of mixture and duct conditions are reported and discussed in detail.

2. Characterization of the gas–particle mixture

2.1. *Thermodynamic properties*

The thermodynamic properties of a mixture of a gas and solid or liquid particles have been analysed by many authors (see e.g. Rudinger 1976). Here only the main results of these analyses will be described, attempting to point out the assumptions under which they may be derived.

In the present work, the following hypotheses are assumed to apply:

(*a*) The mixture is homogeneous, and is composed of a gas phase and a condensed phase in the form of particles uniformly dispersed in the gas.

(*b*) The mixture is chemically inert and there is no mass exchange between the phases.

(*c*) The pressure of the mixture is uniform and equal to that of the gas phase alone.

(*d*) The temperature of the mixture is uniform, i.e. there is no temperature jump between the particles and the gas.

(*e*) The gas phase, which is taken to be incondensible, follows the equation of state of a perfect gas, while the condensed phase is incompressible.

(*f*) The condensed phase is not subject to any change of state, and the calorimetric coefficients of the two phases (and in particular their specific heats at constant pressure and constant volume) are independent not only of pressure and volume, but also of temperature.

Assumptions (*a–d*) characterize the thermodynamic equilibrium of the mixture, while assumptions (*e*) and (*f*) define the behaviour of the two phases. Any state of the mixture may then be identified by means of two of the customary thermodynamic coordinates which are used for single-phase systems, viz. pressure, temperature and density. However, in order to completely describe the properties of the mixture, another variable defining its composition must be introduced. A convenient choice may be the mass fraction, ϕ, defined as the mass of the condensed phase contained in unit mass of the mixture. Indeed, thanks to assumption (*b*), ϕ remains constant during any thermodynamic process of the mixture, considered as a closed system, so that the mixture is again a system defined by two independent coordinates; this is why ϕ may be more appropriate than other variables defining the composition, like the void fraction (i.e. the volume occupied by the gas in unit volume of the mixture) or the concentrations (i.e. the mass of each phase in unit volume of the mixture), which would not be constant.

If ρ_g and ρ_p are the densities of the gas phase and of the particles, respectively, the density of the mixture, ρ_m, may then be obtained from the relation

$$\frac{1-\phi}{\rho_g}+\frac{\phi}{\rho_p}=\frac{1}{\rho_m}. \tag{1}$$

By using assumption (*e*), the equation of state of the mixture becomes

$$p=\frac{\rho_m R_m T}{1-\phi(\rho_m/\rho_p)}, \tag{2}$$

where R_m is defined, as a function of the gas-phase constant, R_g, by

$$R_m=(1-\phi)R_g. \tag{3}$$

As for the specific heats of the mixture, it is easy to obtain

$$C_{p_m}-C_{v_m}=R_m, \tag{4}$$

$$C_{p_m}/C_{v_m}=k_m=k_g\frac{1-\phi+\phi(C/C_{p_g})}{1-\phi+\phi k_g(C/C_{p_g})}, \tag{5}$$

where the suffix g refers to the gas phase, and C is the specific heat of the solid particles.

From (2) it is apparent that when the volumetric fraction of the particles may be neglected (i.e. when $\phi(\rho_m/\rho_p)\ll 1$) the equation of state of the mixture becomes equivalent to that of a 'perfect pseudogas', with a modified gas constant R_m connected with the composition of the mixture through (3) (Wallis 1969; Kieffer 1982).

The differentials of the most important state functions of the mixture, i.e. internal energy, enthalpy and entropy, are then

$$\mathrm{d}u_m=C_{v_m}\,\mathrm{d}T, \tag{6}$$

$$\mathrm{d}h_m=C_{p_m}\,\mathrm{d}T+\phi\frac{\mathrm{d}p}{\rho_p}, \tag{7}$$

$$\mathrm{d}S_m=C_{v_m}\frac{\mathrm{d}T}{T}-R_m\frac{\mathrm{d}\rho_g}{\rho_g}=C_{p_m}\frac{\mathrm{d}T}{T}-R_m\frac{\mathrm{d}p}{p}. \tag{8}$$

The equations describing the main thermodynamic transformations of the mixture may easily be derived from these expressions. In particular, the equation of an isentropic process, as a function of pressure and density, may be written

$$p\left[\frac{1-\phi(\rho_{\mathrm{m}}/\rho_{\mathrm{p}})}{(1-\phi)\,\rho_{\mathrm{m}}}\right]^{k_{\mathrm{m}}} = \text{const.} \tag{9}$$

Another quantity of great interest is the velocity of sound of the mixture. If we assume that the hypothesis of thermodynamic equilibrium of the mixture is strictly satisfied even in presence of small perturbations, then the velocity of sound of the mixture corresponds to

$$a_{\mathrm{m}} = [\partial p/\partial\rho_{\mathrm{m}})_{s_{\mathrm{m}}}]^{\frac{1}{2}} \tag{10}$$

so that the following expression is obtained from (9):

$$a_{\mathrm{m}} = \frac{(k_{\mathrm{m}} R_{\mathrm{m}} T)^{\frac{1}{2}}}{1-\phi(\rho_{\mathrm{m}}/\rho_{\mathrm{p}})}. \tag{11}$$

It is easy to show that the velocity of sound defined by (10) is always lower than that of the pure gas phase at the same conditions, i.e.

$$a_{\mathrm{m}} \leqslant a_{\mathrm{g}} = (k_{\mathrm{g}} R_{\mathrm{g}} T)^{\frac{1}{2}}.$$

The problem of the definition of the velocity of sound in a gas–particle mixture is widely discussed in the literature (see Wallis 1969; Marble 1970; Boothroyd 1971). Indeed, in a generic gas–particle mixture the velocity and the temperature of the particles may differ from those of the gas; as a consequence, four possible definitions, corresponding to different conditions of thermodynamic equilibrium of the mixture, may be given for the velocity of sound. The definition of (10) is then adequate only for perturbations having relatively low frequency, which allow the complete thermomechanical equilibrium to be closely conserved; conversely, for high-frequency perturbations the so-called 'frozen' velocity of sound (i.e. a_{g}, which applies for the gas alone) would be more appropriate. However, for a deeper analysis of the significance of the various velocities of propagation of small perturbations in a gas–particle mixture with different degrees of equilibrium in the heat and momentum transfer between the two phases, reference should be made to Marble (1970).

2.2. *Description of the flow*

The motion of a gas–particle mixture may be studied by means of models of different complexity, but often the simpler ones are those that allow sufficiently general analyses to be carried out, without resorting to complex computer codes for the solution of each particular problem. The present analysis makes use of the homogeneous flow model, and is based on the following assumptions: (i) the flow is steady; (ii) the motion is one-dimensional; (iii) the flow is adiabatic in a constant-area duct; (iv) when vertical, the motion is upwards (i.e. opposite to the gravity force); (v) the particles are everywhere not only in thermal but also in mechanical equilibrium with the gas.

The first three assumptions, together with the compressibility of the gas, are the basis of the classical gasdynamic treatment of the Fanno problem for a single-phase fluid. In that problem the flow is generally horizontal, but a more complete treatment is available (Shapiro 1953) with which assumption (iv) may be taken into account. Finally, assumption (v), which is particularly restrictive, characterizes the homogeneous nature of the two-phase flow, so that it is important to discuss the conditions for its applicability.

The mechanical equilibrium is violated when accelerations and decelerations of the flow are present, and when there are external forces acting selectively on the two phases (e.g. the weight force); only if the motion were uniform could the velocities of the particles and of the gas be exactly the same. Indeed, the particles respond to accelerations and decelerations with an inertia that is different from that of the gas, so that the two phases acquire different velocities. It is exactly this difference which gives rise to the aerodynamic dragging of the particles, which in turn tends to restore the initial mechanical equilibrium. An estimate of the time necessary for restoring the equilibrium of the mixture is given by the aerodynamic response time of a single particle, τ_v (also called velocity relaxation time).

As already pointed out, the mechanical equilibrium may also fail owing to the presence of external forces. In the case of vertical upward flow, the particles must necessarily have a velocity, V_p, lower than that of the gas, V_g, so that the consequent drag force may balance their weight.

The knowledge of the relaxation time, τ_v, and of the slip velocity, $|V_g - V_p|$, allows an appraisal to be carried out of how distant the two-phase flow is from mechanical equilibrium. For instance, if U is a characteristic velocity of the mixture (e.g. the mean velocity) and L is a reference dimension of the system (e.g. the length of the duct), the assumption of mechanical equilibrium will be applicable with increasing accuracy the more the following inequalities are satisfied:

$$\tau_v \ll L/U, \tag{12}$$

$$|V_g - V_p| \ll U. \tag{13}$$

For isolated particles or for very dilute flows the slip velocity is of the order of the velocity of uniform fall of an isolated particle in the gas phase, known as terminal velocity $V_{p\infty}$; this may be evaluated, together with τ_v, through appropriate relations taking into account the particle diameter, the flow conditions and the characteristics of the two phases (Wallis 1969; Rudinger 1976).

In the flow of dense mixtures, however, the particles may be subjected not only to wall–particle interference, but also to at least three types of mutual interaction (excluding electrical or gravitational effects): (i) contact forces, (ii) collisions between the particles, and (iii) fluid dynamic interactions.

Non-impulsive contact actions are due to an incomplete fluidization of the mixture and, as may be found in the literature (see e.g. Wallis 1969), may be neglected when the mixture is completely fluidized, i.e. for void fractions, ϵ, larger than certain limiting values (for instance, for spherical indeformable particles the accepted criterion for fluidization is $\epsilon > 0.4$). Conversely, both the collisions between the particles and the fluid dynamic interactions might rigorously be neglected only for very dilute flows, with high void fractions (say $\epsilon > 0.95$).

Some insight on the transport of dense gas–particle mixtures may be obtained from the literature on high-velocity fluidized beds and pneumatic conveying of particles (see e.g. Capes & Nakamura 1973; Leung & Wiles 1976; Yerushalmi, Turner & Squires 1976; Yerushalmi & Avidan 1985). The experimental investigations in these fields, which were mainly carried out in small-diameter ducts at low gas velocities and at nearly atmospheric pressures, showed that backmixing and slip velocities up to one order of magnitude larger than the terminal velocities of the isolated particles may occur. This seems to be due mainly to duct wall influence and to particle aggregation, with consequent formation of clusters (which increase the virtual diameter of the particulate and its effective terminal velocity). However, there is also evidence (Yerushalmi & Avidan 1985) that a reduction of these effects is connected with increasing velocity, duct diameter and pressure, so that at the conditions typical, for example, of volcanological problems, considerably lower slip velocities might be expected.

In any case, owing to the increase of $V_{p\infty}$ with the diameter of the particles, a limit must be set to the maximum particle size for which an equilibrium flow model may be used.

In summary, adopting a usual approach for the analysis of gas–particle mixtures (Wallis 1969; Boothroyd 1971), in the present work it will be assumed that the effect of particle–wall and particle–particle interactions may be taken into account by means of an appropriate modification of the value of the friction coefficient at the duct walls, and that the mechanical equilibrium between the particles and the gas is fulfilled with sufficient accuracy provided conditions (12) and (13) are satisfied.

Somewhat less complex is the analysis of the validity of the assumption of thermal equilibrium, i.e. that the two phases be at the same temperature. Actually, thermal equilibrium might be rigorously maintained only if the phases were able to exchange heat instantaneously, with negligible temperature gradients. Clearly, this is physically impossible, and, if the compressible phase is subject to variations of velocity and, consequently, of temperature, the thermal equilibrium is broken; however, the difference in temperature causes heat to be exchanged between the phases, so that there is an immediate tendency towards a restoring of the equilibrium condition.

An estimate of the time necessary for equilibrium to be restored is given by the thermal relaxation time, τ_T, which may be estimated with sufficient accuracy by assuming that the particles exchange heat with the gas only by conduction or convection (Rudinger 1976). Therefore, the condition for the correctness of the assumption of thermal equilibrium may be written, by analogy to the mechanical equilibrium case,

$$\tau_T \ll L/U \tag{14}$$

It must be pointed out that τ_T and τ_v are generally of the same order of magnitude, and that, consequently, conditions (12) and (14) are or are not satisfied simultaneously.

3. Equations of motion

The equations of motion of the gas–particle mixture will now be written taking into account all the assumptions discussed in the previous section.

If G_p and G_g are the mass flows per unit area of the condensed and of the gas phases, respectively, we may introduce, as a quantity characterizing the flow, the loading ratio

$$\eta = G_p/G_g. \tag{15}$$

An immediate consequence of the assumed steadiness of the flow is that η is constant along the entire length of the duct. Furthermore, thanks to the assumption of mechanical equilibrium between the gas and the particles, the loading ratio is connected with the mass fraction ϕ through the relation

$$\eta = \phi/(1-\phi). \tag{16}$$

The equations of the one-dimensional, steady, adiabatic, upward vertical flow of an equilibrium gas–particle mixture in a constant-area duct may then be written in differential form as follows.

(*a*) mass balance:

$$\frac{\mathrm{d}\rho_{\mathrm{m}}}{\rho_{\mathrm{m}}}+\frac{\mathrm{d}V}{V}=0; \tag{17}$$

(*b*) momentum balance:

$$\rho_{\mathrm{m}}V\mathrm{d}V+\mathrm{d}p+\rho_{\mathrm{m}}\left(g+\frac{4f}{D}\frac{V^2}{2}\right)\mathrm{d}z=0; \tag{18}$$

(*c*) energy balance:

$$\mathrm{d}h_{\mathrm{m}}+\mathrm{d}\left(\frac{V^2}{2}\right)+g\,\mathrm{d}z=0; \tag{19}$$

where V is the velocity of the mixture, D is the hydraulic diameter of the duct, f its friction coefficient $=\tau_w/(\frac{1}{2}\rho_{\mathrm{m}}V^2)$, and z is the (always positive) coordinate in the direction of motion. As only upward vertical flow is considered, a positive sign was taken in front of the gravity term; should downward motion be of interest, it should be replaced with a negative sign and the following discussion should be modified consequently. Conversely, the case of horizontal flow can immediately be obtained from (18) and (19) by considering the particular case $g=0$.

By introducing now all the constitutive equations that follow from the thermodynamic analysis of the mixture, further differential equations can be derived, which are more suitable for a detailed discussion of the characteristics of the motion. To this end it is useful to introduce the Mach number of the mixture, defined as $M=V/a_{\mathrm{m}}$, so that, after some algebraic manipulation (see Buresti & Casarosa 1987) the following equations may be obtained:

$$\frac{\mathrm{d}V}{V}=\frac{k_{\mathrm{m}}M^2}{2(1-M^2)}\left\{\frac{4f}{D}\left[1+\frac{k_{\mathrm{m}}-1}{k_{\mathrm{m}}}\frac{\eta(\rho_{\mathrm{m}}/\rho_{\mathrm{p}})}{1+\eta(1-\rho_{\mathrm{m}}/\rho_{\mathrm{p}})}\right]+\frac{2g}{k_{\mathrm{m}}V^2}\right\}\mathrm{d}z, \tag{20}$$

$$\frac{\mathrm{d}p}{p}=\frac{1}{2(M^2-1)}\left\{\frac{4f}{D}\frac{k_{\mathrm{m}}M^2(1+\eta)}{1+\eta(1-\rho_{\mathrm{m}}/\rho_{\mathrm{p}})}\left[1+\frac{(k_{\mathrm{m}}-1)M^2(1+\eta)}{1+\eta(1-\rho_{\mathrm{m}}/\rho_{\mathrm{p}})}\right]+\frac{2\rho_{\mathrm{m}}g}{p}\right\}\mathrm{d}z, \tag{21}$$

$$\frac{\mathrm{d}T}{T}=\frac{k_{\mathrm{m}}-1}{2(1-M^2)}\left[\frac{1+\eta}{1+\eta(1-\rho_{\mathrm{m}}/\rho_{\mathrm{p}})}\right]^2\left\{\frac{4fM^2}{D}\left[-k_{\mathrm{m}}M^2+\frac{\eta(\rho_{\mathrm{m}}/\rho_{\mathrm{p}}}{1+\eta}\right]-\frac{2g}{k_{\mathrm{m}}R_{\mathrm{m}}T}\left[\frac{1+\eta(1-\rho_{\mathrm{m}}/\rho_{\mathrm{p}})}{1+\eta}\right]^3\right\}\mathrm{d}z, \tag{22}$$

$$\frac{\mathrm{d}S_{\mathrm{m}}}{C_{p_{\mathrm{m}}}}=\frac{(k_{\mathrm{m}}-1)M^2}{2}\frac{4f}{D}\left[\frac{1+\eta}{1+\eta(1-\rho_{\mathrm{m}}/\rho_{\mathrm{p}})}\right]^2\mathrm{d}z. \tag{23}$$

From these equations, useful indications of the variation of the various quantities along the duct may easily be derived. Indeed, as $k_{\mathrm{m}}>1$ and $\rho_{\mathrm{m}}<\rho_{\mathrm{p}}$ always, we have

	$M<1$	$M>1$
$\mathrm{d}V/\mathrm{d}z$	>0	<0
$\mathrm{d}p/\mathrm{d}z$	<0	>0

Therefore, as is the case for the flow of a perfect gas in a horizontal constant-area duct, the upward vertical flow of an equilibrium gas–particle mixture is subject to the phenomenon of choking, i.e. for any subsonic or supersonic initial Mach number there is a maximum value of the length L of the duct (or, more precisely, of the quantity $4fL/D$) for which a solution is possible, and in that condition the outlet Mach number is equal to 1. Conversely, for a given duct geometry and friction factor, if the ratio of the outlet to the inlet pressures, p_2/p_1, is successively decreased, the mass flow increases up to a maximum, which is reached for a critical value of the ratio p_2/p_1, while further reduction of the outlet pressure does not modify the conditions in the duct.

The discussion of the temperature variation is somewhat more involved. To this end it is useful to introduce the limit temperature

$$T_L = \frac{2gD}{fR_g}\frac{[1+\eta(1-\rho_m/\rho_p)]^3}{[\eta(\rho_m/\rho_p)]^2} \tag{24}$$

and the limit Mach number

$$M_L = \left[\frac{\eta(\rho_m/\rho_p)}{2k_m(1+\eta)}\right]^{\frac{1}{2}}. \tag{25}$$

It should be pointed out that these two quantities are functions of the local conditions of the mixture (as this is the case for ρ_m), so that they vary along the duct.

It is then easy to recognize (Buresti & Casarosa 1987) that when $T < T_L$ we have $\mathrm{d}T/\mathrm{d}z < 0$ for $M < 1$ and $\mathrm{d}T/\mathrm{d}z > 0$ for $M > 1$.

Conversely, if the temperature of the mixture is larger than the limit temperature, i.e. $T > T_L$, then we may identify two Mach numbers

$$M_{L1} = M_L\{1-[1-(T_L/T)]^{\frac{1}{2}}\}^{\frac{1}{2}}, \tag{26}$$

$$M_{L2} = M_L\{1+[1-(T_L/T)]^{\frac{1}{2}}\}^{\frac{1}{2}} \tag{27}$$

satisfying the conditions $\quad 0 \leqslant M_{L1} < M_{L2} < 1$

and such that we have for the temperature gradient along the duct

	$M < M_{L1}$	$M = M_{L1}$	$M_{L1} < M < M_{L2}$	$M = M_{L2}$	$M_{L2} < M < 1$	$M > 1$
$\dfrac{\mathrm{d}T}{\mathrm{d}z}$	< 0	$= 0$	> 0	$= 0$	< 0	> 0

In this case, the possibility of an adiabatic heating in subsonic flow is a rather singular result of the analysis, which derives from the form of the constitutive equations of the mixture. Indeed, a hint that particular conditions may exist for such a behaviour might have been derived from the fact that, as can immediately be seen from (7), the Joule–Thomson coefficient of the mixture ($\mu = (\mathrm{d}T/\mathrm{d}p)_{hm}$) is always negative.

Finally, when the local temperature of the mixture equals the limit temperature, we have

$$T = T_L \Rightarrow M_{L1} = M_{L2} = M_L$$

and the result is again similar to that of the first case, save for the condition $M = M_L$, for which the temperature gradient is zero.

The results obtained for upward vertical motion may be applied to the case of a horizontal duct by putting $g = 0$ in all the relevant relations. As may easily be noticed, the qualitative conclusions on the velocity and pressure gradients remain valid, even if (20) and (21) show that the absolute values of the gradients are lower in the horizontal flow case. As regards the temperature gradient, it may be observed that for a horizontal duct the limit temperature vanishes, i.e. we have

$$g = 0 \Rightarrow T_L = 0; \quad M_{L1} = 0; \quad M_{L2} = M_L \sqrt{2},$$

so that the condition $T > T_L$ is always satisfied and, for low Mach numbers ($0 < M < M_{L2}$), the adiabatic subsonic heating of the flow takes place.

It is interesting to note that this result on the temperature gradient in horizontal motion would not have been found had the volume of the particles been neglected, i.e. if we had assumed $\rho_m/\rho_p = 0$; indeed, it is easy to ascertain that this neglect would lead to the incorrect conclusion that $\mathrm{d}T/\mathrm{d}z$ is always < 0 for $M < 1$.

Finally, (23) demonstrates that the entropy gradient in the direction of motion is always positive, both for subsonic and for supersonic flow; furthermore, it is apparent that it depends only on the friction parameter and on the thermodynamic condition of the mixture, so that its expression is formally the same for horizontal and vertical flow.

Equations (20)–(23) contain, as a particular case, those of the classical Fanno problem for the horizontal flow of a perfect gas (Shapiro 1953), which can be retrieved by putting $g = 0$ and $\eta = 0$, and remembering that in that case the values of ρ_m, a_m, R_m and k_m reduce to those of the gas alone. It may also be interesting to point out again that if the volume of the particles is neglected (i.e. if we put $\rho_p = \infty$) the equations for the mixture are formally identical to those of a perfect gas with thermodynamic properties modified according to the particle loading; obviously, this fact considerably simplifies the treatment of the problem.

It may be expedient to introduce, by differentiating the definition of the Mach number, this additional equation

$$\frac{\mathrm{d}M}{M} = \frac{\mathrm{d}V}{V} - \frac{1}{2}\frac{\mathrm{d}T}{T} - \frac{\eta}{1+\eta}\frac{p}{\rho_p R_m T}\frac{\mathrm{d}\rho_m}{\rho_m}. \tag{28}$$

Following now the classical procedure of Shapiro, the simultaneous equations (17), (20), (21), (22) and (28) may be solved by using the Mach number as the independent variable, and by treating the quantity $4fz/D$ as one of the unknowns; the equations may then be integrated between two generic values of the Mach number. To this end, it is useful to recast the equations in the following form:

$$\frac{\mathrm{d}(\ln \rho_m)}{\mathrm{d}M} = -\frac{\mathrm{d}(\ln V)}{\mathrm{d}M}, \tag{29}$$

$$\frac{\mathrm{d}(4fz/D)}{\mathrm{d}M} = -\frac{\dfrac{\mathrm{d}(\ln p)}{\mathrm{d}M} + \dfrac{k_m M^2(1+\eta)}{1+\eta(1-\rho_m/\rho_p)}\dfrac{\mathrm{d}(\ln V)}{\mathrm{d}M}}{\dfrac{Dg}{4f}\dfrac{\rho_m}{p} + \dfrac{k_m M^2(1+\eta)}{2[1+\eta(1-\rho_m/\rho_p)]}}, \tag{30}$$

$$\frac{\mathrm{d}(\ln T)}{\mathrm{d}M} = -\frac{k_m-1}{k_m}\frac{\eta}{1+\eta(1-\rho_m/\rho_p)}\frac{\rho_m}{\rho_p}\frac{\mathrm{d}(\ln p)}{\mathrm{d}M} - \frac{(k_m-1)M^2(1+\eta)^2}{[1+\eta(1-\rho_m/\rho_p)]^2}\frac{\mathrm{d}(\ln V)}{\mathrm{d}M} - \frac{Dg}{4f}\frac{(k_m-1)}{k_m R_m T}\frac{\mathrm{d}(4fz/D)}{\mathrm{d}M}, \tag{31}$$

$$\frac{\mathrm{d}(\ln p)}{\mathrm{d}M} = \frac{1+\eta}{1+\eta(1-\rho_\mathrm{m}/\rho_\mathrm{p})}\frac{\mathrm{d}(\ln \rho_\mathrm{m})}{\mathrm{d}M} + \frac{\mathrm{d}(\ln T)}{\mathrm{d}M}, \tag{32}$$

$$\frac{\mathrm{d}(\ln V)}{\mathrm{d}M} = \frac{1}{M} + \frac{1}{2}\frac{\mathrm{d}(\ln T)}{\mathrm{d}M} + \frac{\eta}{1+\eta}\frac{p}{\rho_\mathrm{p} R_\mathrm{m} T}\frac{\mathrm{d}(\ln \rho_\mathrm{m})}{\mathrm{d}M}. \tag{33}$$

The main difference between the classical Fanno problem for a perfect gas and the present case is that now it is not possible to manipulate the equations to obtain the variation of each of the unknowns as a function of the Mach number alone; consequently, the conditions along the duct are defined not only by the initial and final values of the Mach number, but also by the initial conditions of the flow in terms of pressure and temperature. The integration of the equations must then be carried out numerically, by giving the inlet values of pressure and temperature, p_i and T_i, together with the loading ratio η and the characteristics of the gas and particle phases. As an output, the variation of all the flow quantities along the duct can be obtained, so that parametric analyses for different geometries and inlet conditions can easily be carried out.

4. Applications

Some examples of application of the model to conditions which may be considered, though not exclusively, of volcanological interest, will now be described..

The set of differential equations (29)–(33) was numerically solved to obtain the upward motion of a gas–particle equilibrium mixture in a constant-area duct. To this end, a computer program based on a multistep Runge–Kutta algorithm was developed. The program may be applied both to subsonic and supersonic flow; however, only applications to cases with initial subsonic Mach number will be discussed. All the data obtained in the various simulations are described in detail by Buresti & Casarosa (1987); here only the most interesting results will be reported and analysed.

To obtain the solution, the following quantities must be specified: the physical properties of the gas phase and of the particles, i.e. R_g, ρ_p, C_{p_g}, C; the friction parameter $4f/D$; the initial pressure and temperature of the mixture; the loading ratio η; the initial Mach number or the length of the duct.

The solution is given in terms of pressure, temperature, density, velocity and Mach number of the mixture along the duct, and corresponds to unit Mach number at its final section, i.e. to 'choked' flow conditions. When the initial Mach number is given, the duct length corresponding to choked flow is obtained from the solution of the set of equations; conversely, when the duct length is fixed, we get the initial Mach number corresponding to subsonic choked flow.

For the various mixtures that were analysed, the density and specific heat of the particles were assumed to be, respectively, $\rho_\mathrm{p} = 2600\ \mathrm{kg/m^3}$ and $C = 837\ \mathrm{J/kg\ K}$. These values are characteristic of the silicious materials typical of volcanic magmas, and were assumed to be independent, to a good approximation, of the temperature of the mixture.

As regards the dimensions of the particles, it was assumed, in accordance with the values reported by Wilson *et al.* (1980), Rose (1987) and Macedonio, Pareschi & Santacroce (1988) for explosive Plinian eruptions, that their mean diameter be prevalently of the order of 0.1 mm or less. The gas phase of the mixtures was taken to be composed either of steam or of pure carbon dioxide; in particular, the first of these gases is normally present in large quantities in the eruptions to which the model may be applied (Kieffer 1982). The behaviour of mixtures of CO_2 and particles was analysed more to study the sensitivity of the solution to the nature of the gas phase than to consider a particular volcanological application.

The specific heats of the gases were evaluated as a function of the initial temperature of the mixture by means of empirical correlations (Van Wylen & Sonntag 1976), and then assumed to be constant all along the duct during the thermodynamic process induced by the motion. This procedure was subsequently validated by the analysis of the calculated variation of the temperature along the duct; indeed, the temperature differences between the initial and final sections of the duct were generally found to be limited to a few tens of degrees. However, if necessary, variations of the specific heats with the temperature along the duct might be introduced in the program without great difficulties.

For mixtures having phases with the physical characteristics described above, calculations were performed for a duct of fixed length with different values of the most important parameters: the initial conditions of the mixture (pressure, temperature and, specially, loading ratio η) and the characteristics of the duct, i.e. its attitude (vertical or horizontal) and the value of $4f/D$.

Special care was taken to verify the fulfillment of the criteria for the acceptability of the assumptions on which the homogeneous flow model is based. As often happens with numerical models, this can only be done *a posteriori*, i.e. when, for instance, the values of the mean velocities along the duct are known from the solution of the problem. We may anticipate that, for the cases that were analysed, the assumption of thermomechanical equilibrium seems to be quite acceptable. Indeed, as shown by Buresti & Casarosa (1987), both the thermal and the mechanical relaxation times of the particles were found to be of the order of hundredths of seconds, while their residence time inside the duct was around ten seconds. Many of the mixtures that were analysed, while always being completely fluidized, cannot be classified as dilute, especially for high loading ratios. Nevertheless, even if the experimental data for the conveying of dense mixtures obtained in laboratory conditions were acritically applied to the analysis of the flow in volcanic conduits, the maximum slip velocities for particle diameters of the order of 0.1 mm would range between 1 % and 2 % of the mean flow velocity. Therefore, taking also into consideration the observations of §2.2 on the difference between the volcanological and the laboratory conditions, the conclusion can be drawn that a one-dimensional homogeneous equilibrium model may be sufficiently accurate for first-order predictions of the flow quantities, provided the obtained values are regarded as suitable cross-sectional averages.

However, the serious objection may be raised that pyroclastic flows are not characterized by dimensional uniformity of the particulate; on the contrary, at least a few large particles are probably present, which may give rise to disturbances and instabilities of the motion and, consequently, to a random unsteadiness of the flow. All these phenomena cannot be predicted by means of a homogeneous flow model; nevertheless, the predicted motion may reasonably be expected to closely correspond to the mean motion of the real flow, particularly if the number of large pyroclasts, and the consequent unsteadiness, is sufficiently low. Obviously, care must be exercised when applying the model to cases in which the effects of inhomogeneity or unsteadiness of the flow may be important.

The following basic condition was chosen as representative of a plausible volcanological application: vertical duct 1200 m in length, homogeneous mixture of particles and H_2O, initial temperature of 850 °C, initial pressure of 33 MPa (approximately corresponding to the lithostatic pressure at the depth of 1200 m); as regards the friction parameter $4f/D$, the assumed basic value was $0.004\ \mathrm{m}^{-1}$. This choice derived from an analysis of possible friction coefficients for two-phase flows (Boothroyd 1971), and from the assumption that the diameter of the volcanic conduit be of the order of 10 m. Starting from this condition, the 'choked' subsonic flow can be evaluated provided the values of the loading ratio are given, and an analysis of the importance of this parameter was then carried out.

In the volcanological literature values of the loading ratio as high as 25 are often considered, starting from the assumption that an estimate of η may be obtained from the void fractions present in ejected pumices. As most pumices probably derive from the liquid present in the interstices between the largest bubbles in the disrupting fluid, this entails the assumption that near the disruption surface the bubbles have the same velocity as the liquid magma. Indeed, before disruption the loading ratio may be roughly defined as $\eta = (\sigma_l V_l)/(\sigma_g V_g)$, where σ_l and σ_g are the concentrations, and V_l and V_g are the cross-sectional average velocities of the liquid magma and of the gas, respectively. If the position of the disruption surface is stationary, the loading ratio is constant all along the duct. Now, as already pointed out, it is reasonable to assume that sufficiently far above the disruption zone the velocities of the small magma particles and of the gas be practically equal, but below it, and particularly near the disruption surface, the velocity of the disrupting bubbles is probably considerably larger than that of the liquid magma, even taking the high viscosity of rhyolites into account. Consequently, the evaluation of the loading ratio from the pumice void fractions is likely to lead to significant overestimates.

If slightly unsteady conditions are considered, i.e. if, for instance, the disruption surface is allowed to slowly migrate up or down, the ratio of the concentrations in the pumices would have an even less immediate relation with the loading ratio. In practice, it does not seem impossible that a real degassing of the liquid magma takes place, producing a gas–particle mixture above the disruption level with a gas mass fraction that is significantly higher that that of the original liquid magma.

Taking also into consideration possible non-volcanological applications, the following values of the loading ratio were then analysed: $\eta = 2, 5, 10, 15$. For these conditions, figures 1–4 describe the solution in terms of variations of pressure, temperature, density and velocity along the duct. As is apparent, the results confirm the predictions deduced from the discussion of (20)–(23). It should be noticed, in particular, that the monotonical decrease of the temperature for all the loading ratios is a direct consequence of the fact that the limit temperature T_L, defined by (24), is much larger than the initial temperature of the mixture for all the cases considered (its minimum value being $T_L = 3688$ K for $\eta = 15$).

For the same values of the loading ratios, figure 5 shows the variation of the void fraction ϵ along the duct; as can be seen, for $\eta = 15$ the initial void fraction is close to the value of 0.75 at which, according to some authors (Sparks 1978; Wilson *et al.* 1980) the disruption of the erupting magma takes place.

To outline concisely the influence on the flow solution of the simultaneous variation of different parameters, it may be useful, rather than giving a detailed description of the motion all along the duct, to refer to global quantities, like the mass flow rate per unit area, and the pressure drop between the initial and the final cross-sections of the duct.

Considering the uncertainty with which the friction coefficient and the diameter of the conduit can be predicted in volcanological applications, the first parameter whose influence may be analysed is the quantity $4f/D$, which was successively given

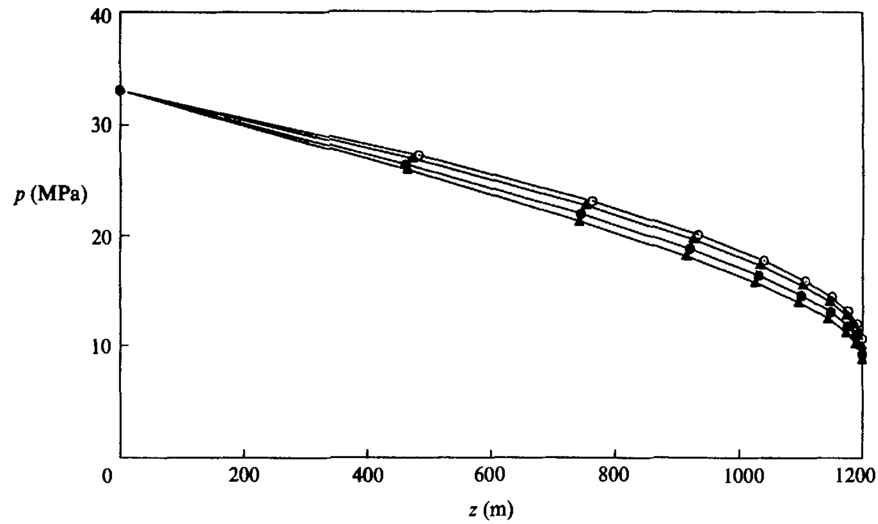

FIGURE 1. Pressure variation along the duct for various loading ratios: ⊙, $\eta = 2$; △, 5; ●, 10; ▲, 15. Vertical choked flow. $L = 1200$ m; gas phase: H_2O; $\rho_p = 2600$ kg m^{-3}; $T_i = 850$ °C; $p_i = 33$ MPa; $4f/D = 0.004$ m^{-1}.

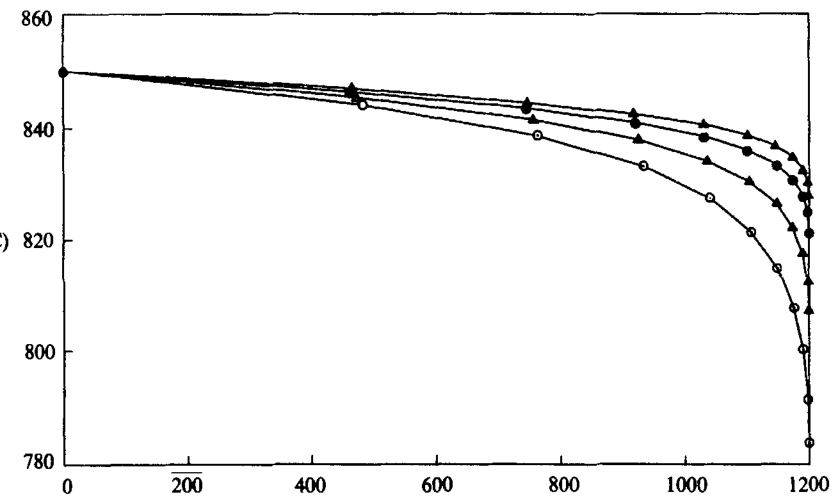

FIGURE 2. Temperature variation along the duct for various loading ratios. Flow conditions and symbols as in figure 1.

the values 0.002, 0.004 and 0.008 m^{-1}. The corresponding results are shown as a function of the loading ratio in figures 6 and 7. As can immediately be noticed, while the mass flow rate has a quite predictable qualitative behaviour, increasing with decreasing friction parameter and with increasing loading ratio, the same is not true for the pressure drop; indeed, this quantity shows a minimum for non-zero values of the loading ratio. In other words, the pressure drop may be lower for choked flows of moderately dilute gas–particle mixtures than for the choked flow of the gas phase alone, at equal values of the initial pressure and of the friction parameter. This result is probably due to the decrease of the velocity of the mixture along the duct

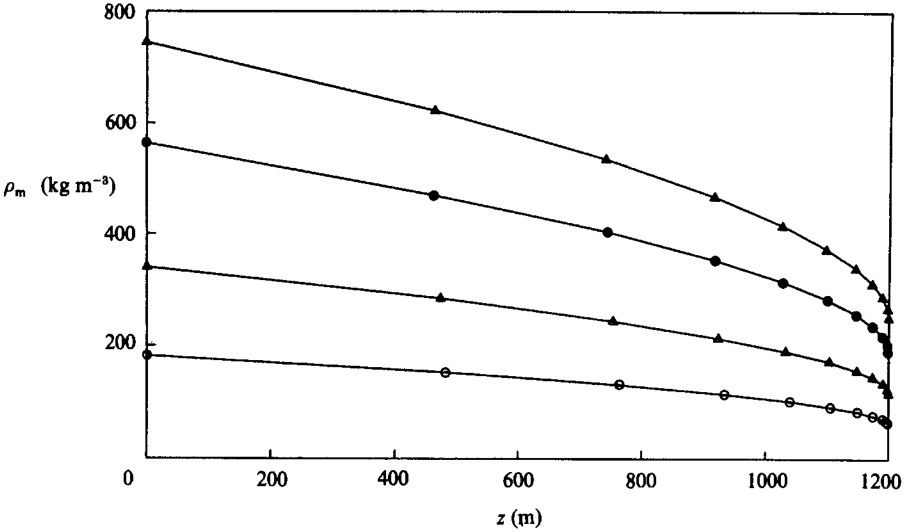

FIGURE 3. Density variation along the duct for various loading ratios. Flow conditions and symbols as in figure 1.

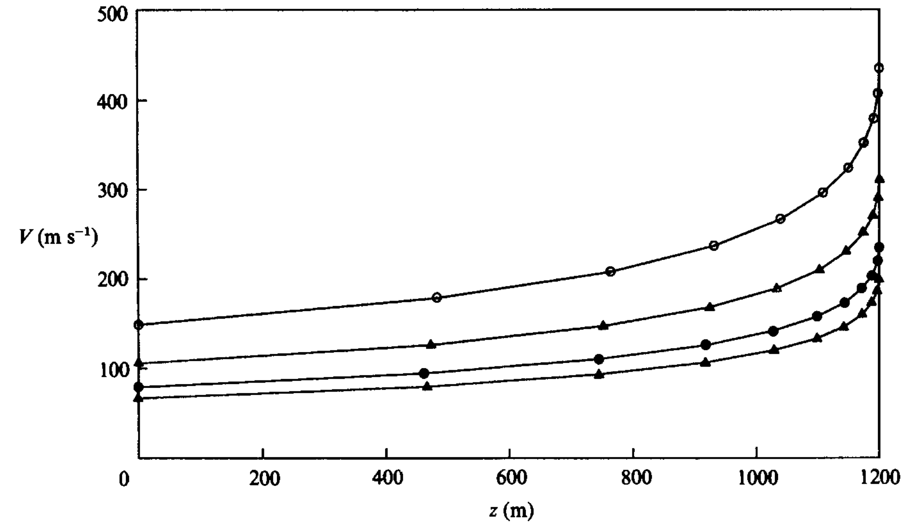

FIGURE 4. Velocity variation along the duct for various loading ratios. Flow conditions and symbols as in figure 1.

(connected with the decrease of the velocity of sound) as η is increased. Indeed, as can be inferred from an analysis of the relevant equations (see Buresti & Casarosa 1987), for small values of the loading ratio this reduction outweighs the increase of the pressure drop due to the increase of η and to gravity; the effect is then expected to be even more pronounced in horizontal than in vertical flow.

Keeping the friction parameter at the constant value of 0.004 m^{-1}, the initial pressure was then varied between 25 and 41 MPa. At equal loading ratios, when the initial pressure is increased both the flow rate and the pressure drop increase as well, and in all cases the qualitative behaviour of the pressure drop as a function of the loading ratio is similar to that already found for an initial pressure of 33 MPa. An

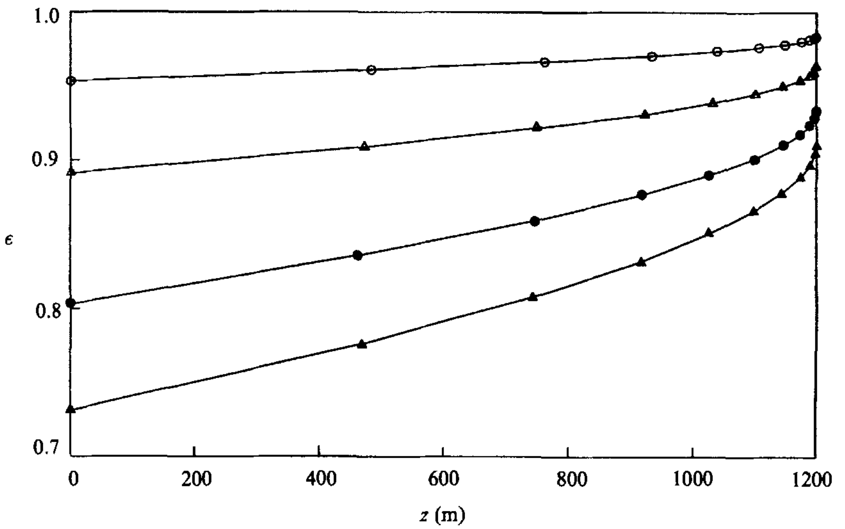

FIGURE 5. Void-fraction variation along the duct for various loading ratios. Flow conditions and symbols as in figure 1.

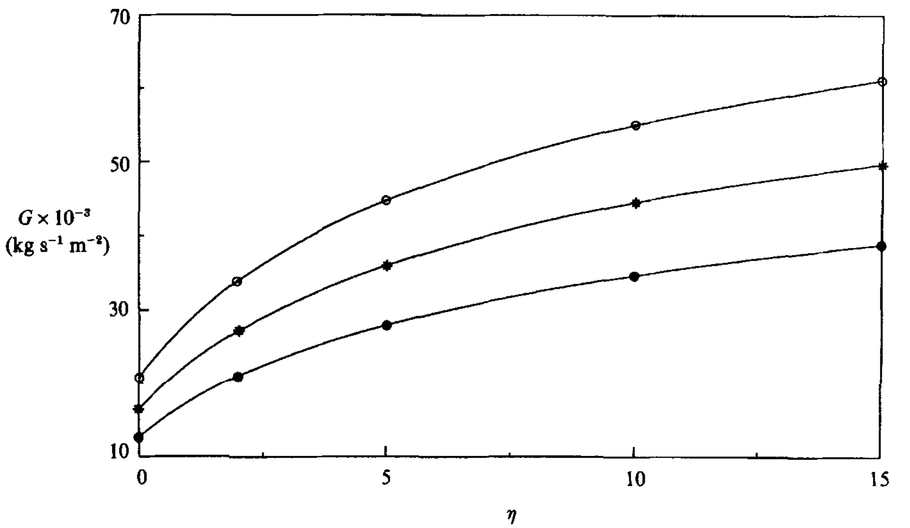

FIGURE 6. Specific mass flow rate *vs.* loading ratio for various values of the friction parameter: $\odot$, $4f/D = 0.002\ \mathrm{m}^{-1}$; *, $0.004\ \mathrm{m}^{-1}$; $\bullet$, $0.008\ \mathrm{m}^{-1}$.

interesting adimensional representation of these results may be given starting from the observation that in the classical Fanno problem for the motion of a pure gas (i.e. for $\eta = 0$) the specific flow rate and the pressure drop are directly proportional to the initial pressure. Figure 8 shows that similar conclusions can be reached for the gas–particle flow as well; indeed, within the range analysed, the ratios between the specific mass flow rate and the initial pressure and between the pressure drop and the initial pressure are almost coincident for the three initial pressures, as the differences, even if apparently not due solely to numerical errors, are certainly very small. This result may lead to considerable simplifications in the

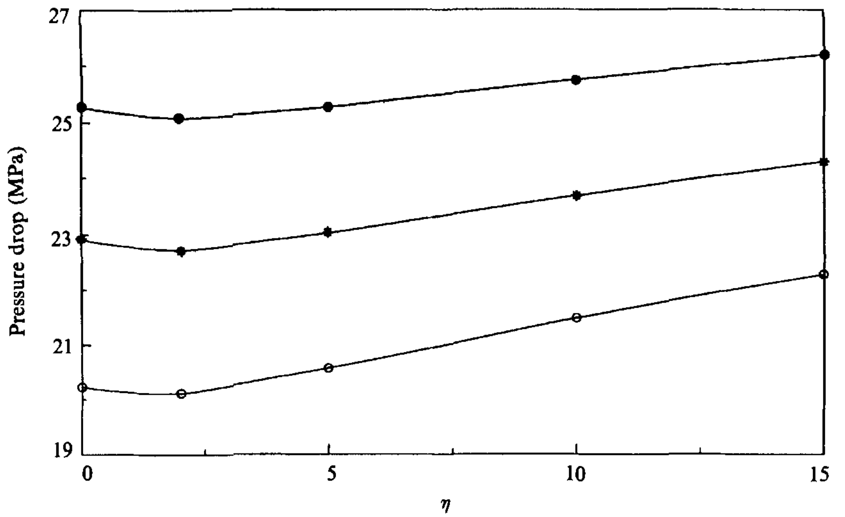

FIGURE 7. Pressure drop along the duct *vs.* loading ratio for various values of the friction parameter. Remaining flow conditions as in figure 1, and symbols as in figure 6.

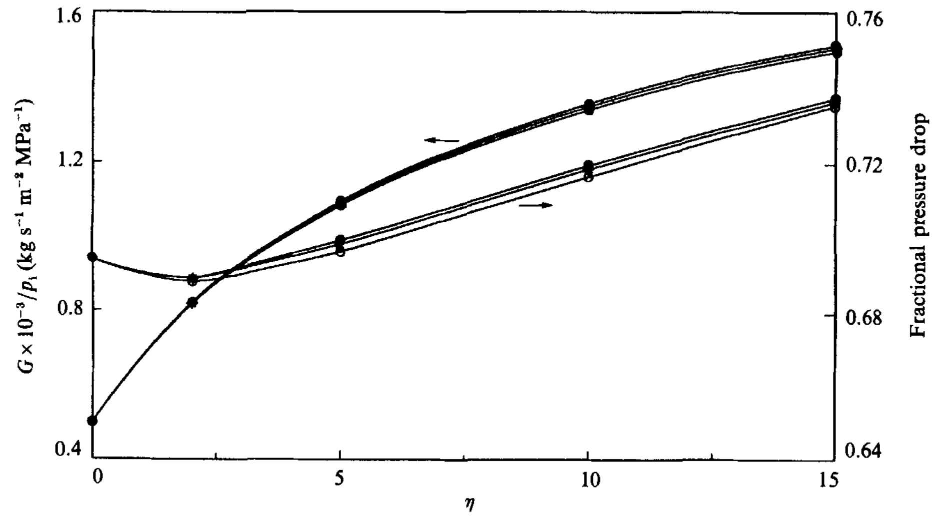

FIGURE 8. Ratio between specific mass flow rate and initial pressure and between pressure drop and initial pressure for various values of the initial pressure: ⊙, $p_i = 25$ MPa; *, 33 MPa; ●, 41 MPa. Remaining flow conditions as in figure 1.

possible application of the model to a quasi-steady analysis of the time-variation of the initial pressure during a volcanic eruption.

It may be interesting, particularly from a fluid dynamic point of view, to analyse the sensitivity of the results to the neglect of the verticality of the flow and of the volume of the particles. The mass flow rate and the pressure drop obtained when these assumptions are made are compared with those of the reference case in figures 9 and 10.

As can be seen, the specific mass flow rate for horizontal flow is always larger than for vertical flow, at least in the range of loading ratios examined; even if not apparent

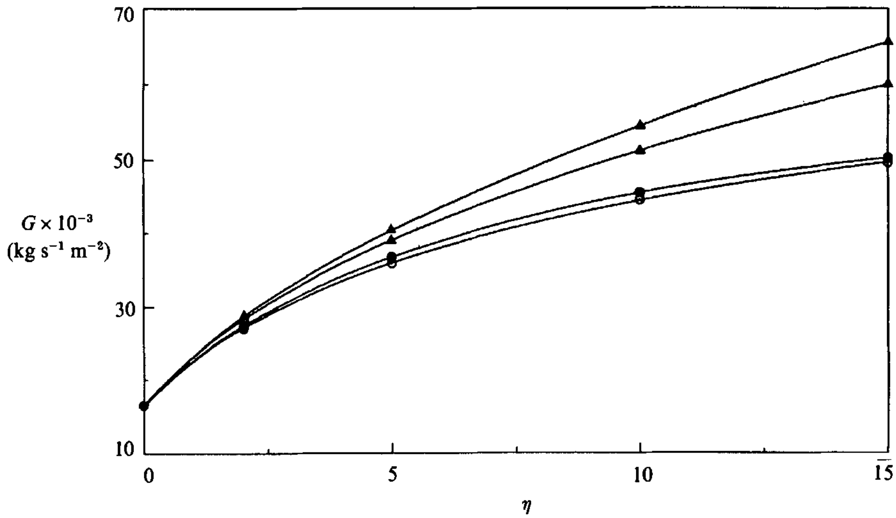

FIGURE 9. Influence of the neglect of verticality and particle volume on the specific mass flow rate: ⊙, vertical; △, horizontal; ●, vertical, $\rho_p = \infty$; ▲, horizontal, $\rho_p = \infty$. Remaining flow conditions as in figure 1.

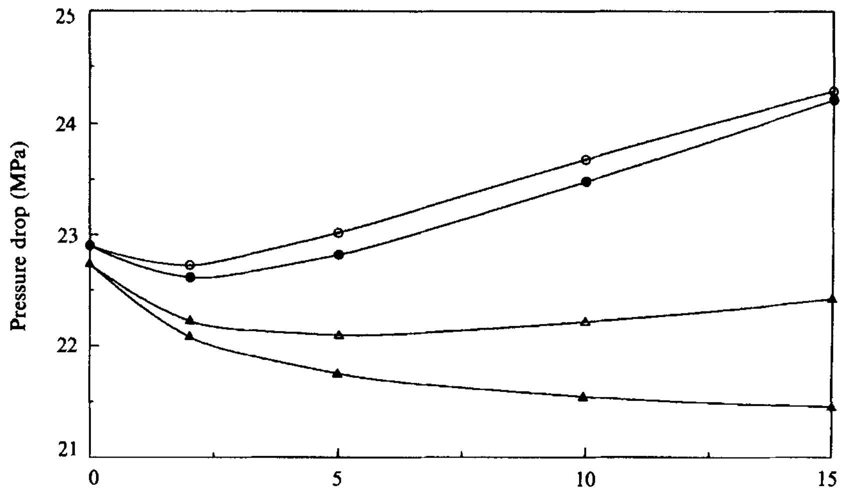

FIGURE 10. Influence of the neglect of verticality and particle volume on the pressure drop along the duct. Remaining flow conditions as in figure 1, and symbols as in figure 9.

from figure 9 owing to scale problems, this is also true for vanishing values of the loading ratio. Obviously, the result obtained for a horizontal duct and $\eta = 0$, i.e. for the pure gas phase, is coincident with that which may be obtained from the classical treatment of the Fanno problem (Shapiro 1953). The neglect of the volume of the particles leads to small overestimates of the specific mass flow rate in the case of vertical flow, while for horizontal flow the errors increase with increasing loading ratio, reaching a value around 10% in the range considered.

As regards figure 10, the pressure drops are always lower for horizontal than for vertical flow. Furthermore, as already anticipated, the minimum in the curve is more evident for horizontal flow, with a considerable increase in the range of loading ratios

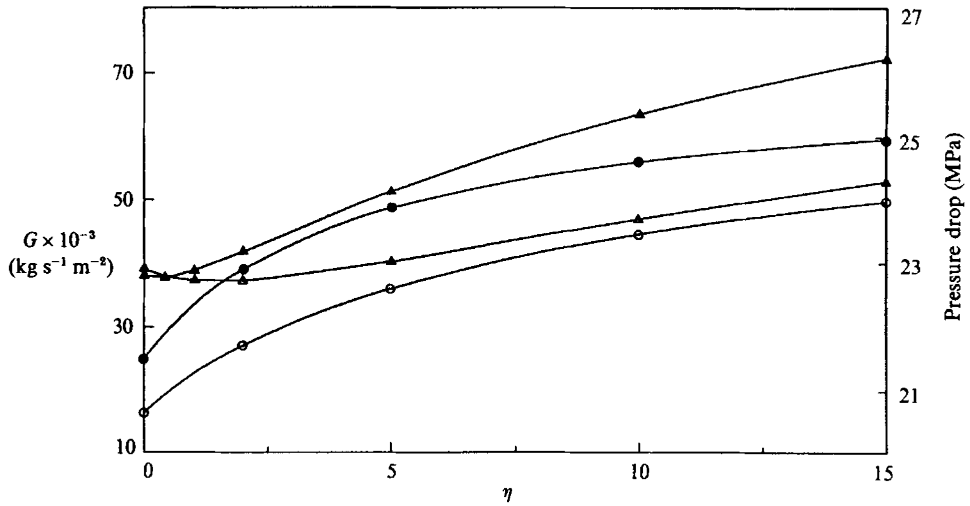

FIGURE 11. Influence of the nature of the gas phase on the specific mass flow rate (○, H_2O; ●, CO_2) and on the pressure drop (△, H_2O; ▲, CO_2) along the duct. Remaining flow conditions as in figure 1.

for which the pressure drop for the mixture is lower than that for the pure gas. When the volume of the particles is neglected, figure 10 shows that the pressure drop is underestimated, with a particularly large error for horizontal flow.

To investigate the sensitivity of the solution to the nature of the gas phase, carbon dioxide was used instead of steam, keeping the characteristics of the particles and the flow conditions unchanged. Figure 11 shows the comparison of the specific mass flow rate and of the pressure drop along the duct obtained with the two gas phases. As is clear, the dependence of the results on the nature of the gas phase is remarkable. Particularly significant is the behaviour of the pressure drop; indeed, for CO_2 the minimum in the curve that was found for steam is still present, but is now confined to very small values of the loading ratio ($\eta < 1$). Furthermore, while for the pure gas phase (i.e. for $\eta = 0$) the pressure drop is lower (even if only slightly) for CO_2 than for H_2O, the situation is reversed for a gas–particle mixture of increasing loading ratio.

This concludes the discussion on the application of the model to situations of volcanological interest. As already pointed out, the numerical results perfectly confirm the variations of the various quantities predicted in the previous section by analysing the structure of (20)–(23). However, the cases examined so far do not correspond to those that might show the peculiar heating of the mixture which was predicted to occur for subsonic adiabatic flow, provided certain requirements on the initial conditions were satisfied. Therefore, it was decided to analyse further cases, not relating to volcanological applications, chosen among those satisfying the aforesaid requirements.

By taking, for instance, a mixture of air and siliceous particles with loading ratios of 2 and 5, at the initial temperature and pressure of 20 °C and 20 MPa, respectively, (24)–(27) give the results shown in table 1 for the initial section of a 100 mm diameter duct ($4f/D = 0.4\ m^{-1}$). In this case, the solution for the choked subsonic flow in a vertical 40 m long duct corresponds, for both values of η, to an initial Mach number lying well between M_{L1} and M_{L2}, so that the temperature along the duct now shows the expected unusual behaviour, documented in figure 12. The same result is obtained if a horizontal duct of the same length is considered.

η	Duct attitude	T_L(°K)	M_{L1}	M_{L2}
2	Vertical	51.84	0.0800	0.3629
2	Horizontal	0	0	0.3716
5	Vertical	13.46	0.0587	0.5413
5	Horizontal	0	0	0.5445

TABLE 1

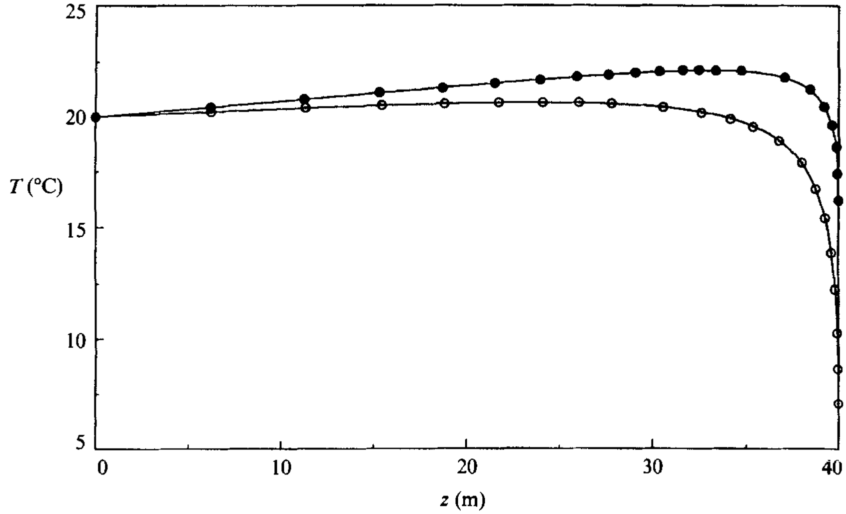

FIGURE 12. Temperature variation along a 40 m duct for two values of the loading ratio: ○, $\eta = 2$; ●, 5. Vertical choked flow. $L = 40$ m; gas phase: air; $\rho_p = 2600$ kg m^{-3}; $T_i = 20$ °C; $P_i = 20$ MPa; $4f/D = 0.4$ m^{-1}.

Significantly different behaviour of the temperature may be found by varying the length of the duct, so that for short ducts (e.g. $L < 8$ m for $\eta = 2$ and $L < 1.9$ m for $\eta = 5$) the adiabatic heating in the subsonic expansion disappears. For very long ducts (e.g. $L \gtrsim 400$ m) the situation becomes more involved, with numerous different cases whose detailed analysis is beyond the scope of the present work.

5. Conclusions

The analysis of the upward motion of gas–particle mixtures in long vertical ducts with friction is one of the many problems suggested by so-called Geological Fluid Mechanics, and stems from the desire to develop a model for the flow of magmatic fluid along volcanic conduits during certain phases of explosive eruptions.

In the present work a simple approach to this problem was taken, i.e. the mixture was assumed to be composed of a perfect gas carrying incompressible particles in conditions of thermomechanical equilibrium; furthermore, the flow was treated as one-dimensional, homogeneous, steady, adiabatic and only constant-area ducts were considered. With all these simplifications, the rigorous deduction of the equations of motion could be carried out, together with an exhaustive analysis of the local variations of the different flow quantities, such as velocity, pressure and temperature. It was then possible not only to show that choked flow may occur, similarly to what happens for a pure perfect gas in ducts with friction, but also to demonstrate the possibility of a singular phenomenon, viz. the adiabatic heating of a mixture undergoing a subsonic expansion.

The flow solutions obtained with the present model are self-consistent and, as a limiting case, contain the classical solution of the Fanno problem for a pure perfect gas. Obviously, the applicability of the model to the interpretation and prediction of physical phenomena (either of volcanological nature or relating to technical problems of different origin) is dependent on the degree of fulfillment of the assumptions on which the model itself is based, and it would be desirable to verify the range of validity of the present analysis through comparison of its results with experimental evidence. Unfortunately, this is not possible owing to the lack of relevant accurate data in the literature; it is hoped that this situation may change in the near future.

As regards the description of eruptive phenomenologies, it is reasonable to maintain that the present analysis may represent a first approximation to the modelling of the flow of fragmentated magmatic fluid (which is characteristic of the Plinian phases of explosive eruptions) in the portion of the volcanic conduit from the disruption level up to the beginning of the crater. The model permits rapid analysis of the influence of the variation of the initial flow conditions (in terms of pressure and temperature at the disruption level), as well as of parameters like conduit diameter, length and wall friction. Furthermore, if a model giving the evolution of the conditions in the liquid, nucleating and disrupting magma as a function of mass outflow is available, the time variation of the flow conditions along the duct may be obtained for the time interval for which quasi-steady flow can be assumed to apply. The analysis can easily be extended to take into account a gradual variability of the cross-section of the conduit, which might significantly alter the choked flow conditions near the exit of the duct.

As for the crater, i.e. the terminal rapidly diverging part of a volcanic conduit, this typical geometric feature may allow the expansion to proceed to supersonic conditions and the exit pressure to drop to values that are nearer to the atmospheric pressure. Owing to the rapid variation of the cross-sectional area, the applicability of the assumption of one-dimensional flow may be more questionable in this region, particularly for gas–particle mixtures with high loading ratios, so that the present mathematical model, even if it were generalized to take the variation of the cross-section into account, might be applied to this portion of the duct only with great caution.

Many further improvements in the model may be envisaged, like the introduction of non-equilibrium between the gas and the particles, or the inclusion of time-dependent phenomena. Anyway, the model should be regarded as but one of the several building blocks which must be available to develop more comprehensive and refined mathematical models tentatively aimed at reproducing the various complex scenarios of volcanic eruptions.

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REFERENCES

Arastoopour, H., Lin, S.-C. & Weil, S. A. 1982 Analysis of vertical pneumatic conveying of solids using multiphase flow models. *AIChE J.* **28**, 467–473.

Barberi, F., Navarro, J. M., Rosi, M., Santacroce, R. & Sbrana, A. 1989 Explosive interaction of magma with ground waters: insights from xenoliths and geothermal drillings. *Rendiconti della SIMP* (In press.)

BOOTHROYD, R. G. 1971 *Flowing Gas-Solid Suspensions*. Chapman and Hall.

BURESTI, G. & CASAROSA, C. 1987 A one-dimensional model for the flow of equilibrium gas-particle mixtures in vertical ducts with friction and its application to volcanological problems. *Atti del Dipartimento di Ingegneria Aerospaziale, ADIA* 87-3, Pisa: ETS Editrice.

CAPES, C. E. & NAKAMURA, K. 1973 Vertical pneumatic conveying: an experimental study with particles in the intermediate and turbulent flow regimes. *Can. J. Chem. Engng* **51**, 31–38.

CROWE, C. T. 1982 Review – Numerical models for dilute gas–particle flows. *Trans. ASME* I: *J. Fluids Engng* **104**, 297–303.

DOSS, E. D. 1985 Analysis and application of solid–gas flow inside a venturi with particle interaction. *Intl J. Multiphase Flow* **11**, 445–458.

HUPPERT, H. E. 1986 The intrusion of fluid mechanics into geology. *J. Fluid Mech.* **173**, 557–594.

KIEFFER, S. W. 1982 Dynamics and thermodynamics of volcanic eruptions: implications for the plumes on Io. In *Satellites of Jupiter* (ed. D. Morrison), pp. 647–723. University of Arizona Press.

LEUNG, L. S. & WILES, R. J. 1976 A quantitative design procedure for vertical pneumatic conveying systems. *Ind. Engng Chem., Process Des. Dev.* **15**, 552–557.

MACEDONIO, G., PARESCHI, M. T. & SANTACROCE, R. 1988 A numerical simulation of the Plinian fall phase of 79 A.D. eruption of Vesuvius. *J. Geophys. Res.* **93**, 14817–14827.

MARBLE, F. E. 1970 Dynamics of dusty gases. *Ann. Rev. Fluid Mech.* **2**, 397–446.

ROSE, W. I. 1987 Interaction of aircraft and explosive eruption clouds: a volcanologist's perspective. *AIAA J.* **25**, 52–58.

RUDINGER, G. 1965 Some effects of finite particle volume on the dynamics of gas-particle mixtures. *AIAA J.* **3**, 1217–1222.

RUDINGER, G. 1970 Gas–particle flow in convergent nozzles at high loading ratios. *AIAA J.* **8**, 1288–1294.

RUDINGER, G. 1976 Fundamentals and applications of gas–particle flow. In *AGARD-AG*-222, pp. 55–86.

SHAPIRO, A. H. 1953 *The Dynamics and Thermodynamics of Compressible Fluid Flow*. Wiley.

SOO, S. L. 1967 *Fluid Dynamics of Multiphase Systems*. Blaisdell, Walthem, Mass.

SPARKS, R. S. J. 1978 The dynamics of bubble formation and growth in magmas: a review and analysis. *J. Volcanol. Geotherm. Res.* **3**, 1–37.

VAN WYLEN, G. J. & SONNTAG, R. E. 1976 *Fundamentals of Classical Thermodynamics*. Wiley.

WALKER, G. P. L. 1981 Plinian eruptions and their products. *Bull. Volcanol.* **44**, 223–240.

WALLIS, G. B. 1969 *One-Dimensional Two-Phase Flow*. MacGraw-Hill.

WILSON, L. & HEAD, J. W. III 1981 Ascent and eruption of basaltic magma on the Earth and Moon. *J. Geophys. Res.* **86**, 2971–3001.

WILSON, L., SPARKS, R. S. J. & WALKER, G. P. L. 1980 Explosive volcanic eruptions – IV. The control of magma properties and conduit geometry on eruption column behaviour. *Geophys. J. R. Astr. Soc.* **63**, 117–148.

YERUSHALMI, J. & AVIDAN, A. 1985 High-velocity fluidization. In *Fluidization*, 2nd Edn (ed. J. F. Davidson, R. Clift & D. Harrison), pp. 225–291, Academic.

YERUSHALMI, J., TURNER, D. H. & SQUIRES, A. M. 1976 The fast fluidized bed. *Ind. Engng Chem., Process Des. Dev.* **15**, 47–53.